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Equation (4) is the differential equation to the curve sought; but, as the second member is not an exact differential, it is doubtful whether the equation can be integrated in finite terms.

If any of our readers will effect an integration of (4), by series or otherwise, so as to obtain the value of x^2 in functions of y suitable for substitution in (3), and will calculate the numerical value of u for any given values of n and r we will insert the discussion, in detail, in No. 6 of the ANALYST.

THE POLYNOMIAL THEOREM.

BY MISS CHRISTINE LADD, POQUONOCK, CONN.

FOR the expansion of the nth power of m quantities the ordinary Algebras offer no better way than to apply the Binomial Theorem $m—1$ times. The process is laborious and the result is not easily brought into a symmetrical form.

Let us write down the sum of m quantities n times, and see how the actual multiplication is performed:

$$\begin{array}{l} a+b+c+d+\;\ldots\, . \\ a+b+c+d+\;\ldots\, . \\ a+b+c+d+\;\ldots\, . \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{array}$$

It is evident that the terms of the product will consist of every possible combination of one or more of the quantities with different exponents in such a manner that the sum of the exponents is equal to n. The only difficulty is to determine the coefficients.

The coefficient of a term expresses the number of ways in which that term can be formed. a^n, b^n &c. can be formed in only one way, therefore their coefficient is 1. The term $a^{n-1}b$ is formed by taking the b of any row and multiplying it by the a of every other row, but as any one of the n b's can be taken there are n terms $a^{n-1}b$, or the coefficient of $a^{n-1}b$ is n; and all terms of the same form, $b^{n-1}d$, $a^{n-1}c, \ldots$ have the same coefficient. To obtain the coefficient of $a^{n-2}b^2$ we observe that the number of ways in which two b's can be selected out of n b's is $\frac{1}{2}n(n—1)$. Take the form $a^{n-5}b^3c^2$. We can take the two c's in $\frac{1}{2}n(n—1)$ ways. Having taken a certain combination of 2 c's we have $n—2$ b's out of which to select 3 b's. This can be done in $\dfrac{(n—2)(n—3)\ldots(n—2—3+1)}{3!}$ ways, and only one combination of the $n-5$ a's remains. Hence the coefficient of $a^{n-5}b^3c^2$ is

$$\frac{n(n-1)}{2!}\cdot\frac{(n-2)(n-3)\dots(n-2-3+1)}{3!}.$$

In general, for the coefficient of

$$a^{n-\beta-\gamma-\delta}b^{\beta}c^{\gamma}d^{\delta}$$

we have the product of the combinations of n things δ at a time by the combinations of $n-\delta$ things γ at a time by the combinations of $n-\delta-\gamma$ things β at a time, which is

$$\frac{n(n-1)\dots(n-\delta+1)}{\delta!}\times\frac{(n-\delta)(n-\delta-1)\dots(n-\delta-\gamma+1)}{\gamma!}$$
$$\times\frac{(n-\delta-\gamma)(n-\delta-\gamma-1)\dots(n-\delta-\gamma-\beta+1)}{\beta!},$$

or

$$\frac{n(n-1)\dots(n-\beta-\gamma-\delta+1)}{\delta!\,.\,\gamma!\,.\,\beta!}.$$

Write $n-\beta-\gamma-\delta=\alpha$ and multiply numerator and denominator of this by $\alpha!$ and we have

$$\frac{n(n-1)\dots(\alpha-1)\alpha(\alpha-1)(\alpha-2)\dots1}{\delta!\,.\,\gamma!\,.\,\beta!\,.\,\alpha!}$$

or

$$\frac{n!}{\alpha!\,.\,\beta!\,.\,\gamma!\,.\,\delta!}$$

for the coefficient of $a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$, in which $\alpha+\beta+\gamma+\delta=n$.

From the manner in which this coefficient is obtained it is evident that all the terms of the form $a^{\beta}c^{\delta}d^{\gamma}b^{\beta}$, $b^{\alpha}c^{\beta}d^{\gamma}e^{\delta}$, and so on, that is to say, all the terms, formed by affecting any four quantities by these four exponents, have the same coefficient.

For example, in the expansion of $(a+b+c+d)^{10}$, the coefficient of $a^4b^3c^2d$ is

$$\frac{10!}{4!\,.\,3!\,.\,2!}=11600.$$

We may then write

$$(a+b+c+\dots)^n=\Sigma\left[\frac{n!}{\alpha!.\beta!.\gamma!\dots}\;\Sigma\left(a^{\alpha}b^{\beta}c^{\gamma}\dots\right)\right],$$

where the Σ within the brackets indicates the sum of the terms formed by all possible combinations of the m quantities with a given set of exponents, and the Σ without the brackets, the sum of all the sets of terms obtained by giving to the exponents all possible values such that

$$\alpha+\beta+\gamma+\dots=n.$$

The number of different coefficients in the expanded form is much less than the number of terms. In the ninth power of nine quantities there are 24310 terms, but only 30 different coefficients.

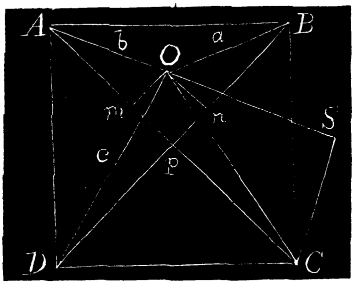
The above demonstration becomes imaginary when n is fractional or negative.

ANSWER TO QUERY AT P. 64, VOL. IV, BY PROF. KERSHNER.— "The extraction of the square root of $a + b\sqrt{-1}$ is an operation to which Euclid's geometry is competent; it requires only the bisection of an angle and the determination of a mean proportional to obtain

$$\left\{ \sqrt[4]{(a^2+b^2)},\ \tfrac{1}{2}\tan^{-1}\frac{b}{a} \right\} \text{ from } \left\{ \sqrt{(a^2+b^2)},\ \tan^{-1}\frac{b}{a} \right\}.$$

Hence it follows that wherever n is a prime number, and $n-1$ is a power of 2, the formation of the nth roots of unity is a geometrical operation, in the ancient sense. This is the discovery of Gauss, and is the most remarkable addition to the power of construction which the ancient geometry has received since the time of Euclid. Euclid mastered the cases $n = 3$, $n = 5$; the next one is $n = 17$, and the next $n = 257$."—See De Morgan's *Trigonometry and Double Algebra.*

GEOMETRICAL SOLUTION OF PROB. 125, BY PROF. W. P. CASEY.—*AO*, *BO* and *DO* are given to find the side of the square.



Join *AC*, *BD* and *CO* and draw *Om*, *On*, perpendiculars, and also *Cs* perpendicular to *AO* produced.

It is well known that $AO^2+CO^2 = DO^2 + BO^2$, i. e., the sum of the squares of the lines drawn from any point to the opposite angles of a square or rectangle = the sum of the squares of the two lines drawn from the same point to the other two opposite angles, and therefore AO^2+CO^2 is given, and AO is given, $\therefore$ CO is a given line.

Again, $DO^2-BO^2 = Dn^2-Bn^2 = 2BD\times pn = 2AC\times Om$. But $2AC \times Om = 2AO\times Cs$, hence $DO^2-BO^2 = 2AO\times Cs$; $\therefore$ $2AO\times Cs$ is given and AO is given, $\therefore$ Cs is given; and CO is given, by the above, $\therefore$ $CO^2 - Cs^2 = Os^2$ is given, and $\therefore$ Os is a given line and AO is given, hence As is given and so is Cs, hence AC is a given line, and $\therefore$ $AC^2 = 2AB^2$ is given and hence AB is a given line.

[The above solution is here given because it is purely geometrical, whereas the solution published at p. 188, Vol. III, contains trigonometric funct's.]